*Int. J. Solids Structures*, 1973, Vol. 9, pp. 141 to 149. Pergamon Press. Printed in Great Britain

# SIMPLER PROBLEMS FOR ELASTIC PLATES

J. L. Ericksen

Mechanics Department, The Johns Hopkins University, Baltimore, Maryland

**Abstract**—Using rather general nonlinear theory, we indicate what are some of the more tractable problems for elastic plates.

## 1. INTRODUCTION

As is discussed by Naghdi [1], variously derived theories of thin elastic plates and shells tend to emerge as special cases of the theory of Cosserat surfaces and it is in this context that we explore the theory of homogeneous flat plates. Leaving the constitutive equations general, we indicate procedures giving rise to simpler problems, interpreted as those involving algebraic or ordinary differential equations. We note some features of such solutions, but do not explore them in depth. The emphasis is on methods of attack, rather than on obtaining end results. In the interests of brevity, discussions of interpretations are kept to a minimum. At the end, we discuss extension of such analyses to right circular cylindrical shells.

## 2. GOVERNING EQUATIONS

Finite deformations convert flat plates to curved shells of a rather general nature, so we begin with the general theory of Cosserat surfaces. These involve a surface, given parametrically by

$$\mathbf{r} = \mathbf{r}(u^\alpha, t),$$

where $\mathbf{r}$ is the position vector from some origin, the $u^\alpha$ are surface material coordinates and $t$ is time. Throughout, commas denote partial derivatives with respect to $u^\alpha$, superposed dots partial derivatives with respect to $t$. Roughly, this can be interpreted as the familiar "middle surface". Associated with it is a director field, a vector field not tangent to it, denoted by

$$\mathbf{d} = \mathbf{d}(u^\alpha, t).$$

Roughly, this represents the relative position vector joining corresponding particles on the top and bottom surfaces of the three-dimensional body envisaged, divided by a suitable fixed measure of shell thickness.

The energy $E$ associated with a subregion is represented by

$$E = \int (W+K)\,\mathrm{d}u^1\,\mathrm{d}u^2, \tag{1}$$

with constitutive equations of the form

$$W = W(\mathbf{r}_{,\alpha}; \mathbf{d}; \mathbf{d}_{,\beta}; u^\gamma) \tag{2}$$

$$2K = \rho(u^\alpha)\dot{\mathbf{r}}\cdot\dot{\mathbf{r}} + 2\rho_1(u^\alpha)\dot{\mathbf{r}}\cdot\dot{\mathbf{d}} + \rho_2(u^\alpha)\dot{\mathbf{d}}\cdot\dot{\mathbf{d}} > 0, \tag{3}$$

the inequality failing only when $\dot{\mathbf{r}} = \dot{\mathbf{d}} = 0$. Commonly, $\rho_1$ is set equal to zero. However, we note that, with a straightforward reinterpretation, Mindlin's [2] calculations, on plated crystal plates, suggest a definite, non-zero value for this case.

Of course, $W$ is to be Galilean invariant,

$$W(R\mathbf{r}_{,\alpha}; R\mathbf{d}; R\mathbf{d}_{,\beta}, u^\alpha) = W(\mathbf{r}_{,\alpha}; \mathbf{d}; \mathbf{d}_{,\beta}; u^\gamma), \tag{4}$$

for all proper orthogonal transformations,

$$R^{-1} = R^T, \qquad \det . R = 1. \tag{5}$$

Equations of motion are of the form

$$\left(\frac{\partial W}{\partial \mathbf{r}_{,\alpha}}\right)_{,\alpha} + \mathbf{f} = \dot{\overline{\frac{\partial K}{\partial \dot{\mathbf{r}}}}}, \tag{6}$$

$$\left(\frac{\partial W}{\partial \mathbf{d}_{,\alpha}}\right)_{,\alpha} - \frac{\partial W}{\partial \mathbf{d}} + \mathbf{g} = \dot{\overline{\frac{\partial K}{\partial \dot{\mathbf{d}}}}}, \tag{7}$$

$\mathbf{f}$ and $\mathbf{g}$ representing loads applied to the surface. From a conventional three-dimensional point of view, they incorporate resultant forces and moments, as well as double forces without moment. If we normalize $W$ to be energy/unit present area or resolve vectors in (6) and (7) into tangential and normal components, we will begin to encounter uses for surface covariant differentiation.

It is convenient to use abbreviated notations suggested by Ericksen [3], writing (6) and (7) as

$$\left(\frac{\partial W}{\partial \mathbf{P}_{,\alpha}}\right)_{,\alpha} - \frac{\partial W}{\partial \mathbf{P}} + \mathbf{F} = \dot{\overline{\frac{\partial K}{\partial \dot{\mathbf{P}}}}} = \mathscr{K}\ddot{\mathbf{P}}, \tag{8}$$

$$\mathbf{P} = (\mathbf{r}, \mathbf{d}), \qquad \mathbf{F} = (\mathbf{f}, \mathbf{q}). \tag{9}$$

That is, $\mathbf{P}$ and other capital bold-face letters denote elements of the six-dimensional vector space $\Sigma$ generated by ordered pairs of vectors in $E_3$. Script capital letters denote linear transformations on $\Sigma$. $\mathscr{K}$ is defined by

$$\mathscr{K}(\mathbf{a}, \mathbf{b}) = (\rho\mathbf{a} + \rho_1\mathbf{b}, \rho_1\mathbf{a} + \rho_2\mathbf{b}). \tag{10}$$

In $\Sigma$, there is the induced inner product, written

$$\{\mathbf{P}_1, \mathbf{P}_2\} = \{(\mathbf{a}_1, \mathbf{b}_1), (\mathbf{a}_2, \mathbf{b}_2)\} \equiv \mathbf{a}_1 \cdot \mathbf{a}_2 + \mathbf{b}_1 \cdot \mathbf{b}_2. \tag{11}$$

Relative to it, $\mathscr{K}$ is symmetric and, from (3), positive definite

$$\mathscr{K} = \mathscr{K}^T > 0. \tag{12}$$

A linear transformation on vectors in $E_3$ induces one on $\Sigma$, for which we use the same letter, e.g.

$$(R\mathbf{a}, R\mathbf{b}) = \mathscr{R}(\mathbf{a}, \mathbf{b}). \tag{13}$$

Of course, not all transformations on $\Sigma$ can be so written, in particular $\mathscr{K}$ cannot, in general. In essence, it is for this reason, as well as brevity, that the abbreviated notation offers some advantages. From (10) and (11), it follows that, when (13) holds,

$$\mathscr{K}\mathscr{R} = \mathscr{R}\mathscr{K}. \tag{14}$$

Other notations should be evident from the chain rule, given the example

$$\left(\frac{\partial W}{\partial \mathbf{P}_{,\alpha}}\right)_{,\alpha} = \frac{\partial^2 W}{\partial \mathbf{P}_{,\alpha}\partial u^\alpha} + \frac{\partial^2 W}{\partial \mathbf{P}_{,\alpha}\partial \mathbf{P}}\mathbf{P}_{,\alpha} + \frac{\partial^2 W}{\partial \mathbf{P}_{,\alpha}\partial \mathbf{P}_{,\beta}}\mathbf{P}_{,\alpha\beta}.$$

## 3. REFORMULATION

We are interested in homogeneous flat plates, for which the obvious reference configuration is of the form

$$\left.\begin{aligned} \mathbf{r} &= \mathbf{r}_R = (u^1, u^2, 0) \\ \mathbf{d} &= \mathbf{d}_R = \text{const.} \end{aligned}\right\} \tag{15}$$

$u^1$ and $u^2$ thus being rectangular Cartesian coordinates. Then, from (1), $W$ and $K$ will represent energies/unit undeformed or reference area. To describe its homogeneity, we restrict $W$ by the condition that

$$W = W(\mathbf{r}_{,\alpha}; \mathbf{d}; \mathbf{d}_{,\beta}; u^\gamma) = W(\mathbf{r}_{,\alpha}; \mathbf{d}; \mathbf{d}_{,\beta}; 0), \tag{16}$$

presuming coordinates chosen so that the origin lies within the plate. We do not require that the reference configuration be a natural state, subject to zero loads, a condition which would further restrict $W$. For our purposes, the additional restrictions are not helpful.

We now make a change of variables, writing

$$\left.\begin{aligned} \mathbf{r} &= \hat{\mathbf{r}} + \hat{R}\mathbf{p}, \\ \mathbf{d} &= \hat{R}\mathbf{q}, \end{aligned}\right\} \tag{17}$$

with the following requirements

$$\left.\begin{aligned} &\hat{R}^T\hat{\mathbf{r}}_{,\alpha} = \mathbf{a}_\alpha = \text{const.} \\ &\hat{R}^T = \hat{R}^{-1}, \qquad \text{det.}\,\hat{R} = 1, \\ &\hat{R}_{,\alpha} = \hat{R}S_\alpha = S_\alpha\hat{R} \\ &S_\alpha = -S_\alpha^T = \text{const.} \end{aligned}\right\} \tag{18}$$

In particular, we could take

$$\hat{\mathbf{r}} = \hat{\mathbf{r}}_R, \qquad \hat{R} = 1, \tag{19}$$

Other possibilities derive from circular cylinders. For example, we can take

$$\hat{r} = a(\cos\psi, \sin\psi, \varphi), \qquad a = \text{const.} \tag{20}$$

where $\varphi$ and $\psi$ are linear functions

$$\varphi = b_\alpha u^\alpha, \qquad \psi = c_\alpha u^\alpha, \tag{21}$$

$$\hat{R} = \begin{Vmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{Vmatrix}, \tag{22}$$

$$S_\alpha = \begin{Vmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Vmatrix} c_\alpha. \tag{23}$$

Here the special case $a = 0$ is not excluded. It can be shown that any $\hat{R}$ which can satisfy (18) can be obtained by transforming this one type to some (three-dimensional) rectangular Cartesian coordinate system. Then $\mathbf{a}_\alpha$ is subject to the condition that

$$S_\beta \mathbf{a}_\alpha$$

be symmetric in $\alpha$ and $\beta$, being otherwise arbitrary, $S_\alpha$ being calculated using $(18)_3$. In all cases, one needs to check that (17) delivers a configuration which could be obtained by deforming, reasonably smoothly, the reference configuration; if we take $\hat{\mathbf{r}} = 0$, we cannot also set $\mathbf{p} = 0$, for example. However, if we use (19) or (20) with $a \neq 0$ and suitable selections (21), we can have $\mathbf{p}$ and $\mathbf{q}$ constant. In the following, we employ fixed $\hat{\mathbf{r}}$ and $\hat{R}$, variable $\mathbf{p}$ and $\mathbf{q}$.

In abbreviated notation, (17) reads

$$\left.\begin{aligned} \mathbf{P} &= (\hat{\mathbf{r}}, 0) + R\mathbf{Q}, \\ \mathbf{Q} &= (\mathbf{p}, \mathbf{q}), \end{aligned}\right\} \tag{24}$$

and

$$\begin{aligned} \mathbf{P}_{,\alpha} &= \hat{\mathscr{R}}\mathbf{T}_\alpha \\ \mathbf{T}_\alpha &= \mathbf{A}_\alpha + \mathscr{S}_\alpha \mathbf{Q} + \mathbf{Q}_{,\alpha} \\ \hat{\mathscr{R}}_{,\alpha} &= \mathscr{S}_\alpha \hat{\mathscr{R}} = \hat{\mathscr{R}} \mathscr{S}_\alpha \\ \mathbf{A}_\alpha &= (\mathbf{a}_\alpha, 0) = \text{const.} \\ \mathscr{S}_\alpha &= \mathscr{S}_\alpha^T = \text{const.} \end{aligned} \tag{25}$$

Recalling that $W$ does not depend explicitly on $\mathbf{r}$ and does satisfy (4), we have

$$\begin{aligned} W(\mathbf{P}, \mathbf{P}_{,\alpha}) &= W(\mathbf{Q}, \mathbf{T}_\alpha), \\ \frac{\partial W}{\partial \mathbf{P}} &= \hat{\mathscr{R}} \frac{\partial W}{\partial \mathbf{Q}}, \\ \frac{\partial W}{\partial \mathbf{P}_{,\alpha}} &= \hat{\mathscr{R}} \frac{\partial W}{\partial \mathbf{T}_\alpha}. \end{aligned} \tag{26}$$

It is convenient to introduce a new function

$$\hat{B}(\mathbf{Q}, \mathbf{Q}_{,\alpha}) = W(Q, \mathbf{T}_\alpha) \tag{27}$$

with $\mathbf{T}_\alpha$ specified as in (25). It will be different for different choices of $\hat{\mathbf{r}}$ and $\hat{R}$, insofar as they give different constants $\mathbf{A}_\alpha$ and $\mathscr{S}_\alpha$, the "hat" serving to remind us of this. A calculation gives

$$\left.\begin{aligned} \frac{\partial \hat{B}}{\partial \mathbf{Q}} &= \frac{\partial W}{\partial \mathbf{Q}} - \mathscr{S}_\alpha \frac{\partial W}{\partial \mathbf{T}_\alpha}, \\ \frac{\partial \hat{B}}{\partial \mathbf{Q}_{,\alpha}} &= \frac{\partial W}{\partial \mathbf{T}_\alpha}. \end{aligned}\right\} \tag{28}$$

Also, since $\hat{\mathbf{r}}$ and $\hat{\mathbf{R}}$ do not depend on $t$ and (14) holds,

$$\mathscr{K}\ddot{\mathbf{P}} = \hat{\mathscr{R}}\mathscr{K}\ddot{\mathbf{Q}}. \tag{29}$$

The equations (8) then become

$$\left.\begin{aligned} \left(\frac{\partial \hat{B}}{\partial \mathbf{Q}_{,\alpha}}\right)_{,\alpha} - \frac{\partial \hat{B}}{\partial \mathbf{Q}} + \mathbf{G} &= \mathscr{K}\ddot{\mathbf{Q}}, \\ \mathbf{G} &= \hat{\mathscr{R}}^{-1}\mathbf{F}, \end{aligned}\right\} \tag{30}$$

where we have used (25), (26), (28) and (29). A consequence of (36), sometimes useful, is

$$\left[\left\{\frac{\partial \hat{B}}{\partial \mathbf{Q}_{,\alpha}}, \mathbf{Q}_{,\beta}\right\} - \hat{B}\delta^\alpha_\beta\right]_{,\alpha} = \{\mathscr{K}\ddot{\mathbf{Q}} - \mathbf{G}, \mathbf{Q}_{,\beta}\}, \tag{31}$$

the curly brackets denoting the inner product (11).

The simplest cases arise when $\mathbf{Q}$ is constant. Then (30) implies that $\mathbf{G}$ must be constant. There are then the algebraic problems of selecting the adjustable constants to make $\mathbf{G}$ have specified properties. If our selection was (20), we might set $\mathbf{p} = 0$ to treat bending of the plate into circular form, for example. If our choice was $\hat{\mathbf{r}} = 0$, $\hat{R} = 1$, we could not set $\mathbf{Q} = \text{const.}$, of course. Some solutions of this variety are discussed by Crochet and Naghdi [3].

Next simplest are the cases where (30) reduces to a system of ordinary differential equations. An obvious possibility is to try

$$\mathbf{Q} = \mathbf{Q}(x), \qquad \mathbf{G} = 0, \qquad x = n_\alpha u^\alpha - Vt \tag{32}$$

where $n_\alpha$ and $V$ are constants. Then (31) gives a first integral,

$$\left\{\frac{\partial \hat{B}}{\partial \mathbf{Q}_{,\alpha}} n_\alpha, \mathbf{Q}'\right\} - \hat{B} = \tfrac{1}{2}\{\mathscr{K}\mathbf{Q}', \mathbf{Q}'\}V^2 + \text{const.} \tag{33}$$

Using (24), (26) and (28)

$$\frac{\partial \hat{B}}{\partial \mathbf{p}} = -S_\alpha \frac{\partial \hat{B}}{\partial \mathbf{p}_{,\alpha}},$$

If (19) applies, $S_\alpha = 0$ and if (23) does,

$$\frac{\partial \hat{B}}{\partial p_3} = 0.$$

Then half of (30), viz.

$$\left(\frac{\partial \hat{B}}{\partial \mathbf{p}_{,\alpha}} n_\alpha\right)' - \frac{\partial \hat{B}}{\partial \mathbf{p}} = V^2(\rho \mathbf{p}'' + \rho_1 \mathbf{q}''),$$

admits three or one easy first integrals. In brief, propagation of such waves in flat or bent plates is relatively tractable. It is of some interest to know when (30) can be uniquely solved for the highest derivatives of $\mathbf{Q}$. These occur linearly, with coefficients

$$\frac{\partial^2 \hat{B}}{\partial \mathbf{Q}_{,\alpha} \partial \mathbf{Q}_{,\beta}} n_\alpha n_\beta - V^2 \mathscr{K} = \mathscr{R}^{-1}\left[\frac{\partial^2 W}{\partial \mathbf{P}_{,\alpha} \partial \mathbf{P}_{,\beta}} n_\alpha n_\beta - V^2 \mathscr{K}\right]\mathscr{R},$$

so they are uniquely determined unless

$$\det.\left\| \frac{\partial^2 \hat{B}}{\partial \mathbf{Q}_{,\alpha} \partial \mathbf{Q}_{,\beta}} n_\alpha n_\beta - V^2 \mathscr{K} \right\| = 0, \tag{34}$$

with the obvious alternative in terms of $W$. As is to be expected, the latter condition agrees with the necessary condition for acceleration waves given in [3]. As a nonlinear effect, the determinant could be zero at only one value of $x$, where we might have an acceleration wave or second derivatives of $\mathbf{Q}$ might become infinite, the commonly recognized symptom of shock wave formation. No doubt, this might be pursued in more depth, but we leave it here.

## 4. LINEARIZED EQUATIONS

*A priori*, it is obvious that linearizing (30) about

$$\mathbf{Q} = \mathbf{Q}_0 = \text{const.}$$

when permissible, will give linear equations with constant coefficients. Also, depending on the choice of $\hat{\mathbf{r}}$ and $\hat{R}$, this linearization can be quite different from that obtaining from linearizing about the reference configuration (15). Here, we use (30) to calculate the constant value of $\mathbf{G}$ required to maintain this "ground state". If you prefer, we adjust $\mathbf{Q}_0$ to suit reasonable demands on $\mathbf{G}$. For the linearization, we write

$$\mathbf{Q} = \mathbf{Q}_0 + \mathbf{V}$$

and, for convenience, assume $\mathbf{G}$ is to be kept at its value in the ground state. Then (30) yields

$$\mathscr{L}^{\alpha\beta}\mathbf{V}_{,\alpha\beta} + \mathscr{L}^{\alpha}\mathbf{V}_{,\alpha} + \mathscr{L}\mathbf{V} = \mathscr{K}\ddot{\mathbf{V}} \tag{35}$$

where the constant linear operators are given by

$$\left.\begin{aligned} \mathscr{L}^{\alpha\beta} &= \frac{\partial^2 \hat{B}}{\partial \mathbf{Q}_{,(\alpha} \partial \mathbf{Q}_{,\beta)}} = \mathscr{L}^{\alpha\beta T}, \\ \mathscr{L}^{\alpha} &= \frac{\partial^2 \hat{B}}{\partial \mathbf{Q}_{,\alpha} \partial \mathbf{Q}} - \frac{\partial^2 \hat{B}}{\partial \mathbf{Q} \partial \mathbf{Q}_{,\alpha}} = -\mathscr{L}^{\alpha T}, \\ \mathscr{L} &= \frac{\partial^2 \hat{B}}{\partial \mathbf{Q} \partial \mathbf{Q}} = \mathscr{L}^{T}, \end{aligned}\right\} \tag{36}$$

all derivatives being evaluated at $\mathbf{V} = 0$. In (36), the parenthesis indicates symmetrization with respect to $\alpha$ and $\beta$. Equations closely analogous to these and comparable in complexity occur in Mindlin's [2] theory of plates.

Of course the simplest solutions are of the type (32),

$$\mathbf{V} = \mathbf{C}\, e^{i(k_\alpha u^\alpha - \omega t)}, \tag{37}$$

with $\mathbf{C}$, $k_\alpha$ and $\omega$ complex constants. We here assume $k_\alpha$ real. Complex $k_\alpha$ can be of interest in connection with "edge waves". Then (35) reduces to

$$\mathscr{H}\mathbf{C} = \omega^2 \mathscr{K}\mathbf{C}, \tag{38}$$

where

$$\mathscr{H} = \mathscr{L}^{\alpha\beta} k_\alpha k_\beta - ik^\alpha \mathscr{L}_\alpha - \mathscr{L} = \mathscr{H}(k_\alpha). \tag{39}$$

From (36) and (39),

$$\mathscr{H}^*(k_\alpha) = \mathscr{H}^T(k_\alpha) = \mathscr{H}(-k_\alpha), \tag{40}$$

so $\mathscr{H}$ is Hermitian, generally not real. It then follows that the six values $\omega^2(k_\alpha)$, obtained by solving

$$\det.(\mathscr{H} - \omega^2 \mathscr{K}) = 0, \tag{41}$$

are real, so $\omega$ is either real or pure imaginary. Also, it follows easily that, if $\mathbf{C}$ and $\mathbf{C}'$ are amplitudes corresponding to different squared frequencies, but the same wave vector,

$$\{\mathbf{C}^*, \mathscr{K}\mathbf{C}'\} = 0. \tag{42}$$

Taking the complex conjugate of (41) and using (40), we see that, with a suitable enumeration of the six branches,

$$\omega^2(k_\alpha) = \omega^2(-k_\alpha). \tag{43}$$

Considering one possibility, with $\omega$ real, one solution (37) generates three others

$$\mathbf{C}\, e^{i(k_\alpha u^\alpha + \omega t)}$$

$$\mathbf{C}^*\, e^{i(-k_\alpha u^\alpha \pm \omega t)}.$$

Adding the four gives a real solution of the form

$$\left.\begin{aligned} \mathbf{V} &= (\mathbf{D} \cos \hat{\varphi} + \mathbf{E} \sin \hat{\varphi}) \cos \omega t, \\ \hat{\varphi} &= k_\alpha u^\alpha. \end{aligned}\right\} \tag{44}$$

Here $\mathbf{D}$ and $\mathbf{E}$ are proportional only if $\mathbf{C}$ can be taken to be real, a rather exceptional event unless the ground state is such that

$$\mathscr{L}^\alpha = 0. \tag{45}$$

In less abbreviated form, (45) reads

$$\left.\begin{aligned}\frac{\partial^2\hat{B}}{\partial\mathbf{p}\partial\mathbf{p}_{,\alpha}} &= \frac{\partial^2\hat{B}}{\partial\mathbf{p}_{,\alpha}\partial\mathbf{p}},\\ \frac{\partial^2\hat{B}}{\partial\mathbf{p}_{,\alpha}\partial\mathbf{q}} &= \frac{\partial^2\hat{B}}{\partial\mathbf{p}\partial\mathbf{q}_{,\alpha}},\\ \frac{\partial^2\hat{B}}{\partial\mathbf{q}_{,\alpha}\partial\mathbf{q}} &= \frac{\partial^2\hat{B}}{\partial\mathbf{q}\partial\mathbf{q}_{,\alpha}}.\end{aligned}\right\} \tag{46}$$

With the choice (19), $(46)_1$ will hold, but the rest need not. With (20), **D** and **E** being constant means that these always bear the same relation to the generators and normal to the cylinder, as well as the direction perpendicular to both. In essence, (44) depicts the phenomenon of acoustical activity, though the underlying mechanism for it need not be of microscopic origin, and these waves are here dispersive, in general. Toupin [5] seems to have been first to discuss three-dimensional theories predicting this phenomenon. In the "high frequency limit",

$$k_\alpha \to \infty, \qquad \omega^2/k_\alpha k_\alpha \to V^2$$

when this makes sense, (41) reduces to (34), giving rise to the common loose analogy between high frequency oscillation and pulse propagation.

## 5. REMARKS

If we look back at the mathematics to see what makes it feasible, it is (16) and (16) alone. Introducing the reference configuration (15) serves an intuitive purpose, but only that. Another case where (16) can be plausible is in bodies having the form of right circular cylinders, referred to "Cartesian coordinates" such as the $(\varphi, \psi)$ occurring in (20), if **d** is of fixed magnitude and makes fixed angles with the generators, normal etc. That is, suppose that, in such a configuration, any two parts of the body are, apart from the translation and rotation required to make them coincide, indistinguishable. Relative to such a reference configuration, (16) should hold. There are only minor differences. For example, in (44), conditions of periodicity will restrict $k_\alpha$. From analyses such as are given by Ericksen [6], it would seem, at best, difficult to make plausible (16) for a third, essentially different area of application.

In the present format, Mindlin's theory can be interpreted as follows. Choose the reference configuration (15) to be one such that

$$\mathbf{d}_R = (0, 0, 1).$$

His is a linear theory of the above type, linearized about the reference configuration. Adopt the identification

$$\mathbf{V} = (\mathbf{u}^{(0)}, \mathbf{u}^{(1)}),$$

after making a minor adjustment to take care of the fact that he chooses coordinates so his second axis is normal to the plate, while our third is. Here $\mathbf{u}^{(0)}$ and $\mathbf{u}^{(1)}$ are the vectors introduced by Mindlin. Inherently, his is a linearization about a very stable natural state, whereas (35) need not be. Whenever it is, the coefficients will have special properties not mentioned here.

*Acknowledgment*—This work was supported by a grant from the National Science Foundation.

## REFERENCES

[1] P. M. NAGHDI, The theory of shells and plates, in *Handbuch der Physik*, to be published.
[2] R. D. MINDLIN, Progress in applied mechanics, *The Prager Anniversary Volume*, pp. 73–84. Macmillan (1963).
[3] J. L. ERICKSEN, *Archs ration. Mech. Analysis* **43**, 167 (1971).
[4] M. J. CROCHET and P. M. NAGHDI, *Int. J. Engng Sci.* **7**, 309 (1969).
[5] R. A. TOUPIN, *Archs ration. Mech. Analysis* **11**, 385 (1962).
[6] J. L. ERICKSEN, *Archs ration. Mech. Analysis* **37**, 73 (1970).

(*Received* 22 *February* 1972; *revised* 5 *June* 1972)

**Абстракт**—Пользуясь даже общей нелинейной теорией, указываются задачи, которые можно разработать более легко.